Student Number: ____________________ **Teacher Name:** ______________________

PENRITH SELECTIVE HIGH SCHOOL
MATHEMATICS DEPARTMENT

2024 TERM 3 TRIAL HSC EXAMINATION

YEAR 12 Mathematics Extension 1

General Instructions:

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided with this examination paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- No correction tape or white out allowed.

Total marks: 70

Section I – 10 marks (pages 1–3)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 4–7)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Topic \ Question	Multiple Choice	Q.11	Q.12	Q.13	Q.14	Total
Combinatorics		b **/ 2**	a **/ 2**			**/ 4**
Functions	3 **/ 1**	a **/ 3**			a **/ 2**	**/ 6**
Proof	2 **/ 1**			b **/ 4**	b **/ 4**	**/ 9**
Trigonometric Functions		e **/ 5**	e **/ 5**			**/ 10**
Calculus	4, 6, 9, 10 **/ 4**		b, c **/ 3+2**	a, c, d **/ 4+3+4**	c **/ 4**	**/ 24**
Vectors	1, 7, 8 **/ 3**	c **/ 2**	d **/ 3**		d **/ 5**	**/ 13**
Binomial Distribution	5 **/ 1**	d **/ 3**				**/ 4**
Total	**/ 10**	**/ 15**	**/ 15**	**/ 15**	**/ 15**	**/ 70**

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 Given that $\overrightarrow{OA} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, What is the direction of $\overrightarrow{AB}$ to the nearest degree?

A. $37°$

B. $53°$

C. $-53°$

D. $143°$

2 Mathematical induction is a method of proof that can be used to prove a statement:

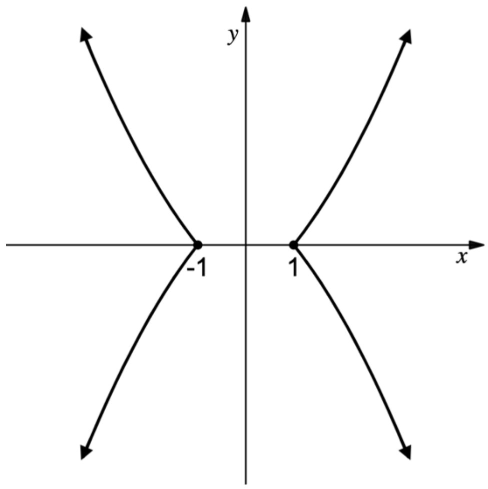
A. only for all positive integers n.

B. for all integers $n \geq$ any fixed integer.

C. for all real numbers n.

D. for all integers n.

3 For $f(x) = (x + 3)(x - 1)$, which one of the following represents the graph below?



A. $|y| = f(x)$

B. $y = |f(|x|)|$

C. $|y| = |f(x)|$

D. $|y| = f(|x|)$

4 Which of the following integrals produces the volume of a cone?

A. $\pi \int_0^2 x\, dx$

B. $\pi \int_{-1}^2 (y^2 - 1)^2\, dy$

C. $\pi \int_{-1}^2 (2 - x)^2\, dx$

D. $\pi \int_{-2}^0 (2y - 2)^2\, dy$

5 Find $P(X = 3)$ if $X \sim B\left(4, \frac{5}{8}\right)$.

A. $\binom{4}{3}\left(\frac{5}{8}\right)^3\left(\frac{3}{8}\right)$

B. $1 - \binom{4}{3}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^3$

C. $\left(\frac{5}{8}\right)^3$

D. $\binom{4}{3}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^3$

6 Which of the following is **not** a first-order linear differential equation?

A. $xy' + x^2y = 1$

B. $y \sin x - y' \cos x = 0$

C. $y'y + 4x = 2$

D. $y' = 3x^2$

7 A particle is projected with initial speed V ms^{-1} at an angle of α to the horizontal.
If its position at t seconds is given by $\underset{\sim}{r} = 40t\mathbf{i} + \left(9t - \frac{1}{2}gt^2\right)\mathbf{j}$, which of the following statements is correct?

A. $V = 7\ \text{ms}^{-1}$

B. $V = \dfrac{\sqrt{6481}}{2}\ \text{ms}^{-1}$

C. $V = \sqrt{1681 - g^2}\ \text{ms}^{-1}$

D. $V = 41\ \text{ms}^{-1}$

8 For non-zero vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$, what is $\text{proj}_{3\underset{\sim}{q}} 6\underset{\sim}{p}$ if $\text{proj}_{\underset{\sim}{q}} \underset{\sim}{p} = \underset{\sim}{r}$?

A. $\underset{\sim}{r}$

B. $2\underset{\sim}{r}$

C. $6\underset{\sim}{r}$

D. $18\underset{\sim}{r}$

9 Find $\displaystyle\int \tan x \sec^2 x \, dx$.

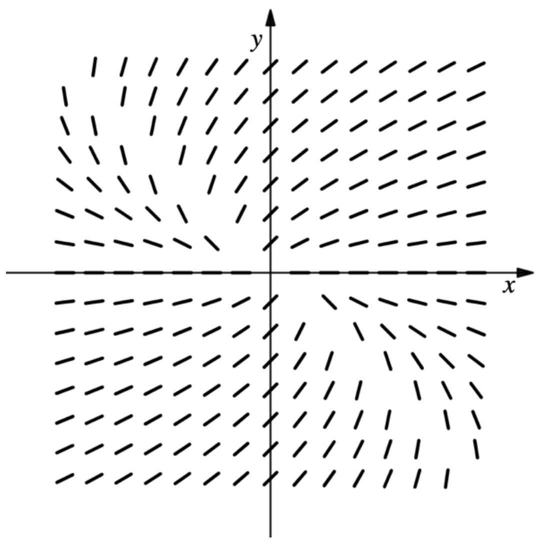
A. $\sec x + C$

B. $\dfrac{1}{2}\tan^2 x + C$

C. $\dfrac{\sin x}{\cos^3 x} + C$

D. $\sec^2 x + C$

10 Which differential equation is a possible match with the slope field presented below?



A. $y' = \dfrac{x}{x+y}$

B. $y' = \dfrac{y}{x-y}$

C. $y' = \dfrac{x}{x-y}$

D. $y' = \dfrac{y}{x+y}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section.

- Answer the questions in the spaces provided.
- Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate Writing Booklet

(a) Solve $\dfrac{4-x^2}{x-1} \leq 0$. **3**

(b) A committee of 7 is to be chosen from 6 men and 8 women. How many different committees can be formed if the committee must have exactly 4 men? **2**

(c) Three forces $\underset{\sim}{F}_1 = 80$ N at 290°T, $\underset{\sim}{F}_2 = 110$ N at 040°T and $\underset{\sim}{F}_3 = 90$ N at 180°T are acting on an object. Show that the resultant force $\underset{\sim}{F}$ is $-4.5\mathbf{i} + 21.6\mathbf{j}$, corrected to one decimal place. **2**

(d) A binomial random variable, X, has $\mathrm{E}(X) = \dfrac{2}{3}$ and $\mathrm{Var}(X) = \dfrac{5}{9}$. Calculate $P(X \geq 1)$. **3**

(e) (i) Show that the equation $3\cos x + 2\sin x = -3$ can be written as $2t + 3 = 0$, where $t = \tan\dfrac{x}{2}$. **2**

(ii) Hence solve the equation for $0 \leq x \leq 2\pi$, correct to 1 decimal place where necessary. **3**

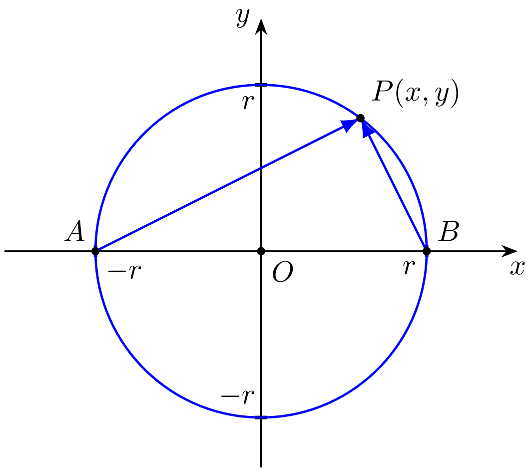
Question 12 (15 marks) Use a separate Writing Booklet

(a) Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{n} = \overrightarrow{ON}$. Find $\text{proj}_{\underset{\sim}{n}}\underset{\sim}{a}$ as a multiple of $\underset{\sim}{n}$ if $|OA| = 4$, $|ON| = 6$ and $\angle AON = 30°$. **2**

(b) Calculate the area of the region bounded by the curves $y = x^2 - 6$ and $y = x$, given that they intersect at $(3, 3)$ and $(-2, -2)$. **3**

(c) Find $\displaystyle\int \frac{\ln x}{x}\,dx$, using the substitution $u = \ln x$. **2**

(d) The graph of a circle with radius r and centre at $(0, 0)$ is shown below. Prove that $\overrightarrow{AP}$ and $\overrightarrow{BP}$ in the diagram are perpendicular using vectors. **3**



(e) (i) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ using the expansion of $\cos(2\theta + \theta)$. **2**

(ii) Hence solve $8x^3 - 6x - \sqrt{3} = 0$ using the substitution $x = \cos\theta$. **3**

Question 13 (15 marks) Use a separate Writing Booklet

(a) (i) Explain why $y = \sin^{-1} x + \cos^{-1} x$ is a constant function. **2**

(ii) Hence find the constant. **2**

(b) Trevor states that $n^2 + 3n$ is an odd integer for all integers $n \geq 1$.

(i) Show that the statement is true for $n = k + 1$ if it is true for $n = k$, where k is an integer greater than or equal to 1. **2**

(ii) Is Trevor's statement true? Justify your answer. **2**

(c) A solid is formed by rotating the region bounded by the curve $y = 6 - x^2$ and $y = 2$ about the y-axis. Find the exact value of the volume of the solid. **3**

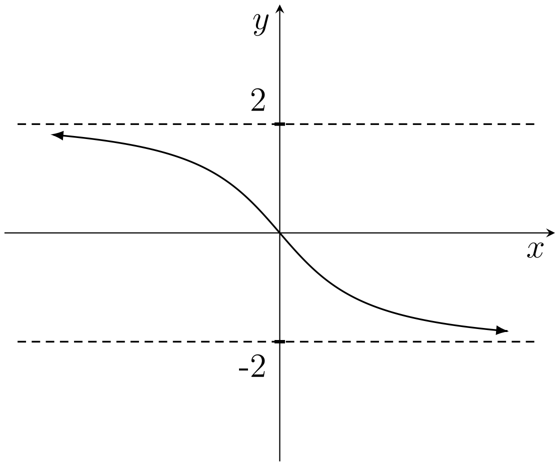
(d) Consider the differential equation $\dfrac{dy}{dx} = xe^{-y}$.

(i) Explain why the differential equation does **not** have a constant solution. **2**

(ii) Find the other solutions of the differential equation by separating the variables. **2**

Question 14 (15 marks) Use a separate Writing Booklet

(a) Given the graph of $y = f(x)$ below, sketch $y = \dfrac{1}{f(x)}$, showing the key features clearly. **2**



(b) Prove by mathematical induction that $8(8^n - 1) - 7n$ is divisible by 49 for all integers $n \geq 0$. **4**

(c) A rabbit population of 500 was released on an island. The population growth is modelled by the logistic equation $\dfrac{dP}{dt} = \dfrac{P}{10}\left(1 - \dfrac{P}{2000}\right)$. **4**

Given that $\dfrac{20000}{P(2000 - P)} = 10\left(\dfrac{1}{P} + \dfrac{1}{2000 - P}\right)$, solve the differential equation to show that the population P at time t months after introduction is $P = \dfrac{2000}{1 + 3e^{-\frac{t}{10}}}$.

(d) A stone is projected from level ground with initial speed V ms^{-1} at an angle of θ to the horizontal. The maximum height reached by the stone was 8 metres.

(i) By integrating vectors, show that the velocity and displacement of the stone at t seconds are as below. **2**

$$\underset{\sim}{v} = V\cos\theta\,\mathbf{i} + (V\sin\theta - gt)\mathbf{j}$$
$$\underset{\sim}{r} = Vt\cos\theta\,\mathbf{i} + \left(Vt\sin\theta - \frac{g}{2}t^2\right)\mathbf{j}$$

(ii) Prove that the horizontal range of the stone is $\sqrt{\dfrac{64}{g}(V^2 - 16g)}$. **3**

End of paper

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Functions	3 **/ 1**	a **/ 3**			a **/ 2**	**/ 6**
Proof	2 **/ 1**			b **/ 4**	b **/ 4**	**/ 9**
Trigonometric Functions		e **/ 5**	e **/ 5**			**/ 10**
Calculus	4, 6, 9, 10 **/ 4**		b, c **/ 3+2**	a, c, d **/ 4+3+4**	c **/ 4**	**/ 24**
Vectors	1, 7, 8 **/ 3**	c **/ 2**	d **/ 3**		d **/ 5**	**/ 13**
Binomial Distribution	5 **/ 1**	d **/ 3**				**/ 4**
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Section I

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Use the multiple-choice answer sheet for Questions 1–10.

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1 Given that $\overrightarrow{OA} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, What is the direction of $\overrightarrow{AB}$ to the nearest degree?

A. $37°$

B. $53°$

C. $-53°$

D. $143°$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} -8 \\ 6 \end{bmatrix}$

$\tan\theta = \dfrac{6}{8}$

$\theta = \tan^{-1}\dfrac{6}{8}$

$\theta = 37°$ (nearest degree)

$\therefore$ Since $(-8, 6)$ lies in the 2nd quadrant, the direction of $\overrightarrow{AB}$ is $180° - 37° = 143°$.

Answer: D

2 Mathematical induction is a method of proof that can be used to prove a statement:

A. only for all positive integers n.

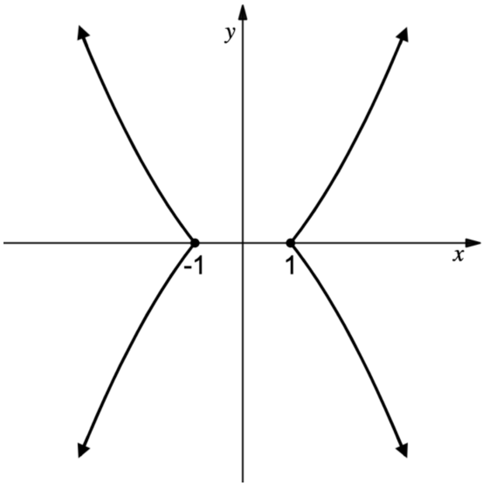
B. for all integers $n \geq$ any fixed integer.

C. for all real numbers n.

D. for all integers n.

Answer: A or B

3 For $f(x) = (x+3)(x-1)$, which one of the following represents the graph below?



A. $|y| = f(x)$

B. $y = |f(|x|)|$

C. $|y| = |f(x)|$

D. $|y| = f(|x|)$

Answer: D

4 Which of the following integrals produces the volume of a cone?

A. $\pi \int_0^2 x \, dx$

B. $\pi \int_{-1}^2 (y^2 - 1)^2 \, dy$

C. $\pi \int_{-1}^2 (2 - x)^2 \, dx$

D. $\pi \int_{-2}^0 (2y - 2)^2 \, dy$

Answer: C

5 Find $P(X=3)$ if $X \sim B\left(4, \frac{5}{8}\right)$.

A. $\binom{4}{3}\left(\frac{5}{8}\right)^3\left(\frac{3}{8}\right)$

B. $1-\binom{4}{3}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^3$

C. $\left(\frac{5}{8}\right)^3$

D. $\binom{4}{3}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^3$

Answer: A

6 Which of the following is **not** a first-order linear differential equation?

A. $xy' + x^2y = 1$

B. $y\sin x - y'\cos x = 0$

C. $y'y + 4x = 2$

D. $y' = 3x^2$

A first-order differential equation is called linear if it can be put into the form
$y' + f(x)y = g(x)$, where $f(x)$ and $g(x)$ functions of x.

Option A is $y' + xy = \frac{1}{x}$

Option B is $y' - y\tan x = 0$

Option C can be rearranged as $y' + \frac{4x}{y} = \frac{2}{y}$ which is not in the required form.

Option D has the y term missing but it is still in the required form. It is a case when $f(x) = 0$.

Answer: **C**

7 A particle is projected with initial speed V ms^{-1} at an angle of α to the horizontal.

If its position at t seconds is given by $\underset{\sim}{r} = 40t\mathbf{i} + \left(9t - \frac{1}{2}gt^2\right)\mathbf{j}$, which of the following statements is correct?

A. $V = 7$ ms^{-1}

B. $V = \dfrac{\sqrt{6481}}{2}$ ms^{-1}

C. $V = \sqrt{1681 - g^2}$ ms^{-1}

D. $V = 41$ ms^{-1}

$\underset{\sim}{v} = 40\mathbf{i} + (9 - gt)\mathbf{j}$

When $t = 0$, $\quad \underset{\sim}{v} = 40\mathbf{i} + 9\mathbf{j}$

Initial speed $= \sqrt{40^2 + 9^2}$

$= \sqrt{1681}$

$= 41$ ms^{-1}

Answer: D

8 For non-zero vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$, what is $\text{proj}_{3\underset{\sim}{q}} 6\underset{\sim}{p}$ if $\text{proj}_{\underset{\sim}{q}}\underset{\sim}{p} = \underset{\sim}{r}$?

A. $\underset{\sim}{r}$

B. $2\underset{\sim}{r}$

C. $6\underset{\sim}{r}$

D. $18\underset{\sim}{r}$

Answer: C

9 Find $\displaystyle\int \tan x \sec^2 x \, dx$.

A. $\sec x + C$

B. $\dfrac{1}{2}\tan^2 x + C$

C. $\dfrac{\sin x}{\cos^3 x} + C$

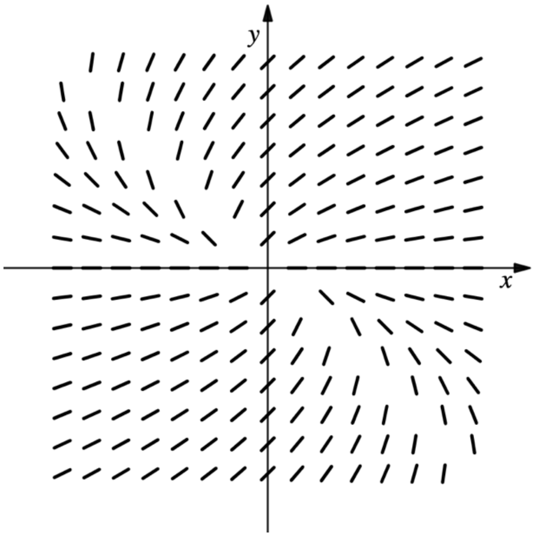
D. $\sec^2 x + C$

$$\int \sec^2 x\,(\tan x)\,dx = \frac{(\tan x)^2}{2} + C \qquad \text{(reverse chain rule)}$$

$$= \frac{1}{2}\tan^2 x + C$$

Answer: B

10 Which differential equation is a possible match with the slope field presented below?



A. $y' = \dfrac{x}{x+y}$

B. $y' = \dfrac{y}{x-y}$

C. $y' = \dfrac{x}{x-y}$

D. $y' = \dfrac{y}{x+y}$

- The slopes are undefined for the points where x and y values have the same magnitude but opposite signs. That is, when $x + y = 0$. This excludes option B and C.
- The slopes for the points on the x-axis are zero. That is, $y' = 0$ when $y = 0$. This excludes A.

Answer: **D**

Question 11 (15 marks) Use a separate Writing Booklet

(a) Solve $\frac{4-x^2}{x-1} \le 0$. **3**

$\frac{4-x^2}{\cancel{x-1}} \times (x-1)^{\cancel{2}} \le 0 \times (x-1)^2$, where $x \ne 1$

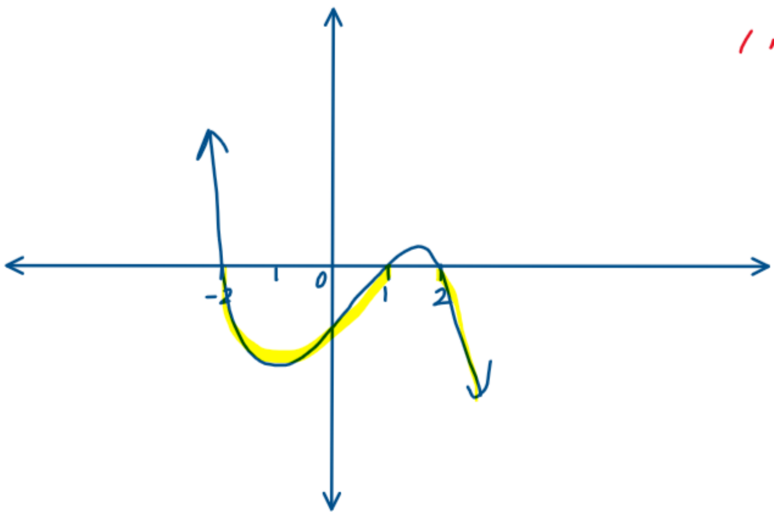
$(4-x^2)(x-1) \le 0$

$(2-x)(2+x)(x-1) \le 0$

1 mark for multiplying $(x-1)^2$ both sides

1 mark for $x \ne 1$

1 mark for correct solution



Comment: most students did not state $x \ne 1$.

$\therefore -2 \le x < 1$ or $x \ge 2$

(b) A committee of 7 is to be chosen from 6 men and 8 women. How many different committees can be formed if the committee must have exactly 4 men? **2**

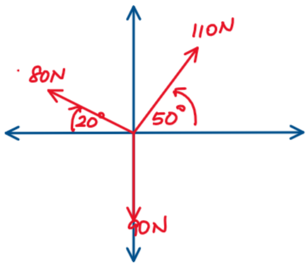
${}^6C_4 \times {}^8C_3 = 840$

1 mark for each

2 marks for correct answer.

Well done

(c) Three forces $\underset{\sim}{F}_1 = 80$ N at 290°T, $\underset{\sim}{F}_2 = 110$ N at 040°T and $\underset{\sim}{F}_3 = 90$ N at 180°T are acting on an object. Show that the resultant force $\underset{\sim}{F}$ is $-4.5\mathbf{i} + 21.6\mathbf{j}$, corrected to one decimal place. **2**



$\underset{\sim}{F}_1 = -80\cos 20°\,\underset{\sim}{i} + 80\sin 20°\,\underset{\sim}{j}$

$\underset{\sim}{F}_2 = 110\cos 50°\,\underset{\sim}{i} + 110\sin 50°\,\underset{\sim}{j}$

$\underset{\sim}{F}_3 = -90\,\underset{\sim}{j}$

$\underset{\sim}{F} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$

$= (-80\cos 20° + 110\cos 50°)\,\underset{\sim}{i} + (80\sin 20° + 110\sin 50° - 90)\,\underset{\sim}{j}$

$= -4.5\,\underset{\sim}{i} + 21.6\,\underset{\sim}{j}$

Not well done.
Most students mixed sin and cos and used 70° or 40°.

(d) A binomial random variable, X, has $E(X) = \frac{2}{3}$ and $Var(X) = \frac{5}{9}$. Calculate $P(X \geq 1)$. **3**

$E(X) = np = \frac{2}{3}$ —— ①

$Var(X) = npq = \frac{5}{9}$ —— ②

Sub ① into ②

$\frac{2}{3}q = \frac{5}{9}$

$q = \frac{5}{\cancel{9}_3} \times \frac{\cancel{3}^1}{2}$

$= \frac{5}{6}$

$p = 1 - q$

$= 1 - \frac{5}{6}$

$p = \frac{1}{6}$

Sub $p = \frac{1}{6}$ into ①

$n\left(\frac{1}{6}\right) = \frac{2}{3}$

$n = \frac{2}{\cancel{3}_1} \times \frac{\cancel{6}^2}{1}$

$n = 4$

$P(X \geq 1) = 1 - P(X=0)$

$= 1 - \left({}^4C_0 \times \left(\frac{1}{6}\right)^0 \times \left(\frac{5}{6}\right)^4\right)$

$= 1 - \frac{625}{1296}$

$= \frac{671}{1296}$

2 marks for solving simultaneously and finding n, p, q.

1 mark for correct answer

(e) (i) Show that the equation $3\cos x + 2\sin x = -3$ can be written as $2t + 3 = 0$, where $t = \tan\frac{x}{2}$. **2**

$$\frac{3(1-t^2)}{1+t^2} + \frac{2(2t)}{1+t^2} = -3$$

$$3 - 3t^2 + 4t = -3(1+t^2)$$

$$3 - \cancel{3t^2} + 4t = -3 - \cancel{3t^2}$$

$$4t + 6 = 0$$

$$2(2t+3) = 0$$

$$2t + 3 = 0$$

Well done

(ii) Hence solve the equation for $0 \le x \le 2\pi$, correct to 1 decimal place where necessary. **3**

$$2t + 3 = 0$$

$$2t = -3$$

$$t = -\frac{3}{2}$$

$\tan\frac{x}{2} = -\frac{3}{2}$ for $0 \le \frac{x}{2} \le \pi$

related angle $\frac{x}{2} = \tan^{-1}\left(\frac{3}{2}\right)$ tan is negative in quad 2.

$\frac{x}{2} = 0.9827937232$ 1 mark for finding the related angle

Since $\tan\frac{x}{2} < 0$, $\frac{x}{2}$ lies in 2nd quadrant

$$\frac{x}{2} = \pi - 0.9827937232$$

$$= 2.15879893$$

$\therefore x = 4.317597861$

$= 4.3$ (1.d.p) 1 mark for 4.3

Check: when $x = \pi$ (since $t = \tan\frac{x}{2}$)

$LHS = 3\cos\pi + 2\sin\pi$

$= -3 + 0$

$= -3$

$= RHS$

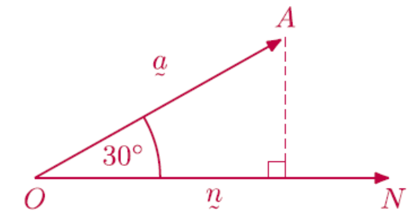
$\therefore x = \pi$ is a solution 1 mark for checking $x = \pi$ and conclude it is a solution.

$\therefore x = 4.3$ (1.d.p) or π

Question 12 (15 marks) Use a separate Writing Booklet

(a) Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{n} = \overrightarrow{ON}$. Find $\text{proj}_{\underset{\sim}{n}}\underset{\sim}{a}$ as a multiple of $\underset{\sim}{n}$ if $|OA| = 4$, $|ON| = 6$ and $\angle AON = 30°$. **2**

Solution 1:



$$\text{proj}_{\underset{\sim}{n}}\underset{\sim}{a} = (|\underset{\sim}{a}|\cos 30°)\,\hat{\underset{\sim}{n}}$$ ⇒ **1 mark**

$$= 4 \times \frac{\sqrt{3}}{2} \times \frac{\underset{\sim}{n}}{6}$$
$$= \frac{\sqrt{3}}{3}\underset{\sim}{n}$$
} **1 mark**

Solution 2:

$$\underset{\sim}{a} = \begin{bmatrix} 4\cos 30° \\ 4\sin 30° \end{bmatrix}, \qquad \underset{\sim}{n} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2\sqrt{3} \\ 2 \end{bmatrix}$$
$$\text{proj}_{\underset{\sim}{n}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{n}}{\underset{\sim}{n}\cdot\underset{\sim}{n}}\underset{\sim}{n}$$
$$= \frac{12\sqrt{3}+0}{36+0}\underset{\sim}{n}$$
$$= \frac{\sqrt{3}}{3}\underset{\sim}{n}$$

Marker's Comment:

Common error is not using the correct formula for the projection formula. Some students used this incorrect formula with the wrong denominator:

$$\text{proj}_{\underset{\sim}{n}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{n}}{\underset{\sim}{a}\cdot\underset{\sim}{a}}\underset{\sim}{n}$$

However, this question was mostly well done.

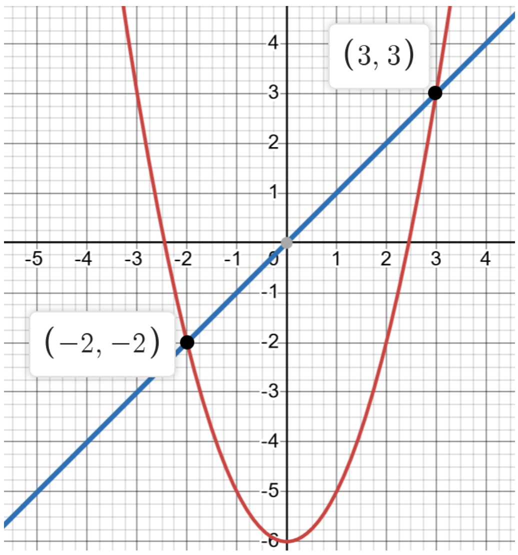
(b) Calculate the area of the region bounded by the curves $y = x^2 - 6$ and $y = x$, given that they intersect at $(3, 3)$ and $(-2, -2)$. **3**

$$A = \int_{-2}^{3} \left(x - (x^2 - 6)\right) dx$$ ⇒ **1 mark**

$$= \int_{-2}^{3} (x - x^2 + 6)\, dx$$
$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 6x\right]_{-2}^{3}$$ ⇒ **1 mark**

$$= \left(\frac{3^2}{2} - \frac{3^3}{3} + 6(3)\right) - \left(\frac{(-2)^2}{2} - \frac{(-2)^3}{3} + 6(-2)\right)$$ ⇒ **1 mark**

$$= \frac{125}{6}\ \text{unit}^2$$



Marker's Comment:

It is important that student's show the line of substitution. Always show the substitution of the upper limit and lower limit. You may be awarded marks even though you made a calculation error on your calculator.

This question was mostly well done.

(c) Find $\int \frac{\ln x}{x} dx$, using the substitution $u = \ln x$. **2**

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x} dx$$

$$\left.\begin{aligned}\int \frac{\ln x}{x} dx &= \int \frac{1}{x} \ln x \, dx \\ &= \int u \, du \\ &= \frac{u^2}{2} + C\end{aligned}\right\} \text{1 mark}$$

$$= \frac{1}{2}(\ln x)^2 + C \Longrightarrow \text{1 mark}$$

Marker's Comment:

Most common error is interpreting $(\ln x)^2$ as $\ln x^2$.

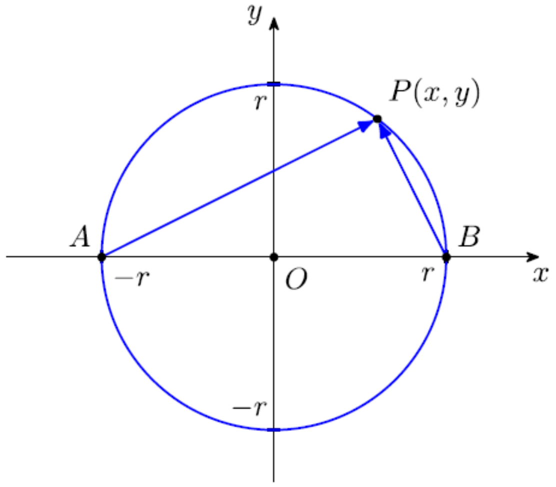
$(\ln x)^2 = (\ln x)(\ln x)$

$\ln x^2 = 2\ln x$

Some students forgot to substitute back the $\ln x$ on to *u*.

However, this question was mostly well done.

(d) The graph of a circle with radius r and centre at $(0, 0)$ is shown below. **3**
Prove that $\overrightarrow{AP}$ and $\overrightarrow{BP}$ in the diagram are perpendicular using vectors.

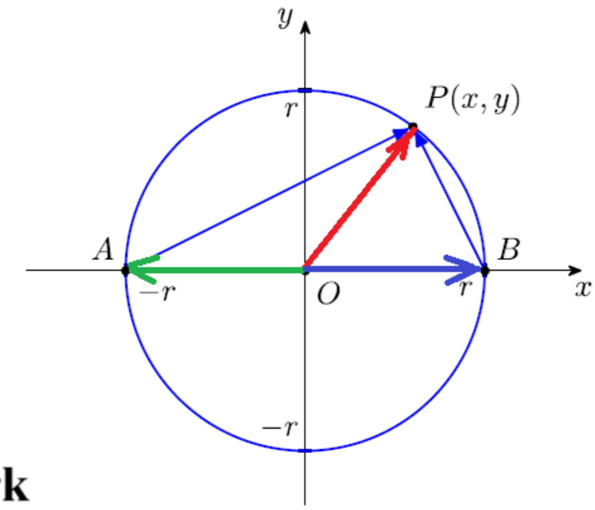


Required to show: $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$ $\left(\cos 90^\circ = 0\right)$

$$\begin{aligned}
\overrightarrow{AP} &= \overrightarrow{OP} - \overrightarrow{OA} \\
&= \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -r \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} x + r \\ y \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BP} &= \overrightarrow{OP} - \overrightarrow{OB} \\
&= \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} r \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} x - r \\ y \end{bmatrix}
\end{aligned}$$

1 mark



$$\begin{aligned}
\overrightarrow{AP} \cdot \overrightarrow{BP} &= \begin{bmatrix} x + r \\ y \end{bmatrix} \cdot \begin{bmatrix} x - r \\ y \end{bmatrix} \\
&= (x + r)(x - r) + y^2 \\
&= x^2 - r^2 + y^2 \\
&= x^2 + y^2 - r^2 \\
&= r^2 - r^2 \\
&= 0
\end{aligned}$$

1 mark

Since $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$

$\therefore$ $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are perpendicular.

1 mark

Marker's Comment:

No marks are awarded for students who did not use vectors to prove that AP and BP are perpendicular.

So far, this section was not too bad.

(e) (i) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ using the expansion of $\cos(2\theta+\theta)$. **2**

$$\begin{aligned}
\text{LHS} &= \cos 3\theta \\
&= \cos(2\theta+\theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \\
&= \text{RHS}
\end{aligned}$$

1 mark

1 mark

Marker's Comment:

Mostly well done.

(ii) Hence solve $8x^3 - 6x - \sqrt{3} = 0$ using the substitution $x = \cos\theta$. **3**

Substitute $x = \cos\theta$,

$$8\cos^3\theta - 6\cos\theta - \sqrt{3} = 0$$

$$8\cos^3\theta - 6\cos\theta = \sqrt{3}$$

$$4\cos^3\theta - 3\cos\theta = \frac{\sqrt{3}}{2}$$

$$\cos 3\theta = \frac{\sqrt{3}}{2} \qquad \text{(from (i))}$$

1 mark

$$\text{Related angle} = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

Since $\cos 3\theta > 0$, 3θ lies in the 1st and 4th quadrants.

$$3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \dots$$

1 mark

$$\theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \dots$$

$$x = \cos\frac{\pi}{18}, \cos\frac{11\pi}{18}, \cos\frac{13\pi}{18}$$

1 mark

Marker's Comment:

Most students lost mark by not listing more than 3 values. It is important that students list more than 3 although it is a cubic polynomial since we are looking for distinct roots and verify that they are distinct values. Some students forgot to solve for x eventually, thus losing marks again.

Question 13 (15 marks) Use a separate Writing Booklet

(a) (i) Explain why $y = \sin^{-1} x + \cos^{-1} x$ is a constant function. **2**

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$
$$= 0$$

1 mark

$\therefore$ The function is constant since its derivative is zero. 1 mark

If you have used a graphical method, you must provide a detailed explanation to earn 2 marks.

(ii) Hence find the constant. **2**

$\sin^{-1} x + \cos^{-1} x = c$ (from (a))

Substitute $x = 0$, (since y is constant, choose any x value from the domain of $-1 \le x \le 1$)

$c = \sin^{-1} 0 + \cos^{-1} 0$ 1 mark

$= 0 + \frac{\pi}{2}$

$\therefore c = \frac{\pi}{2}$ 1 mark

'Hence ...'
Penalised for any other methods.

(b) Trevor states that $n^2 + 3n$ is an odd integer for all integers $n \ge 1$.

(i) Show that the statement is true for $n = k + 1$ if it is true for $n = k$, where k is an integer greater than or equal to 1. **2**

Assume true for $n = k$, where k is an integer.
i.e. $k^2 + 3k = 2P + 1$, where P is an integer.

Prove true for $n = k + 1$.
i.e. $(k+1)^2 + 3(k+1) = 2Q + 1$, where Q is an integer.

LHS $= k^2 + 2k + 1 + 3k + 3$
$= (2P + 1 - 3k) + 5k + 4$ (by the assumption) 1 mark
$= 2P + 2k + 5$
$= 2(P + k + 2) + 1$
$= 2Q + 1$
$=$ RHS
$\therefore$ The statement is true for $n = k + 1$ if it is true for $n = k$. 1 mark

Most common error:
$\mathbf{3(k+1) \neq 3k+1}$

(ii) Is Trevor's statement true? Justify your answer. **2**

Prove true for $n = 1$, 1 mark
$1^2 + 3(1) = 4$, which is not odd.
Since Trevor's statement is false for $n = 1$, his statement of not true. 1 mark

Must answer the questions to get the 2nd mark.

Question 13 (continues)

(c) A solid is formed by rotating the region bounded by the curve $y = 6 - x^2$ and $y = 2$ about the y-axis. Find the exact value of the volume of the solid. **3**

$x^2 = 6 - y$

$V = \pi \int_2^6 x^2 \, dy$

$= \pi \int_2^6 (6 - y) \, dy$ 1 mark

$= \pi \left[6y - \frac{y^2}{2}\right]_2^6$

$= \pi \left[\left(6(6) - \frac{6^2}{2}\right) - \left(6(2) - \frac{2^2}{2}\right)\right]$ 1 mark

$= 8\pi$ unit^3 1 mark

Poorly done.

Common errors:
- Missing π.
- Incorrect limits.
- Integrating $(6 - y)^2$

The substitution step must be shown.
'Fundamental theorem of calculus.'

(d) Consider the differential equation $\frac{dy}{dx} = xe^{-y}$.

(i) Explain why the differential equation does **not** have a constant solution. **2**

Substitute $\frac{dy}{dx} = 0$ for the constant solution,
$0 = xe^{-y}$ 1 mark

$e^{-y} = 0$
$\therefore$ y is undefined since e^{-y} cannot be zero.
$\therefore$ The constant solution does not exist. 1 mark

(ii) Find the other solutions of the differential equation by separating the variables. **2**

$\frac{dy}{dx} = xe^{-y}$
$e^y dy = x \, dx$
Integrating both sides,
$\int e^y \, dy = \int x \, dx$ 1 mark

$e^y = \frac{x^2}{2} + C$
$y = \ln\left|\frac{x^2}{2} + C\right|$ 1 mark

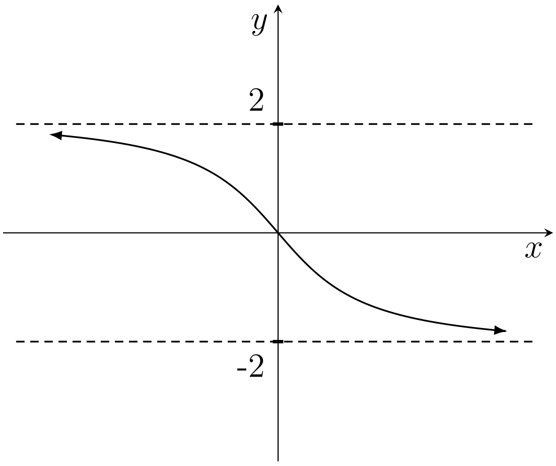
Common errors:

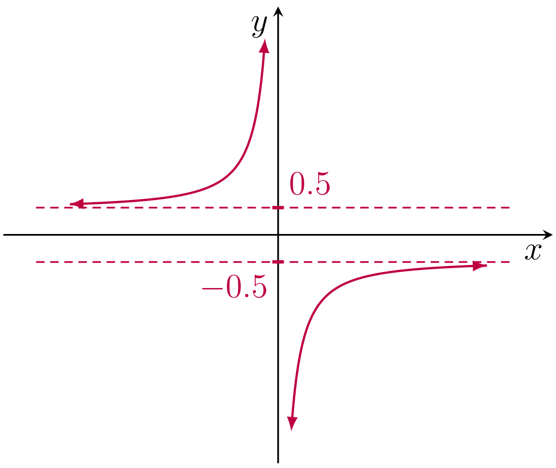
$$\ln\left(\frac{x^2}{2} + C\right) \neq \ln\left(\frac{x^2}{2}\right) + C$$
$$\neq \ln\left(\frac{x^2}{2}\right) + \ln C$$

2024 Feedback – Question 14

Question 14 (15 marks) Use a separate Writing Booklet

(a) Given the graph of $y = f(x)$ below, sketch $= \dfrac{1}{f(x)}$, showing the key features clearly. **2**





Common Mistakes:

- Some students wrote incorrect equations for the asymptotes.
- Some students drew the graph within the asymptotes

- Overall, well done

Learning Strategies:
Students need to

- use a ruler to draw number plane,
- sketch smooth curves,
- pay attention to the key features such as x-intercepts, y-intercepts, asymptotes etc.
- Always label a coordinate on the curve

(b) Prove by mathematical induction that $8(8^n - 1) - 7n$ is divisible by 49 for all integers $n \geq 0$. **4**

Step 1 Prove true for $n = 0$.

$8(8^0 - 1) - 0 = 0,$ which is divisible by 49.

$\therefore$ The statement is true for $n = 0$.

Step 2 Assume true for $n = k$, where k is an integer $k \geq 0$.

i.e. assume $8(8^k - 1) - 7k = 49P$, where P is an integer.

$$8(8^k) - 8 - 7k = 49P$$
$$8(8^k) = 49P + 7k + 8$$

Now prove true for $n = k + 1$.

i.e. prove $8(8^{k+1} - 1) - 7(k + 1)$ is divisible by 49.

$$\begin{aligned}\text{LHS} &= 8(8^{k+1} - 1) - 7(k + 1)\\ &= 8(8^k \cdot 8 - 1) - 7k - 7\\ &= 8(49P + 7k + 8 - 1) - 7k - 7 \quad \text{(by the assumption)}\\ &= 8(49P) + 56k + 56 - 7k - 7\\ &= 8(49P) + 49k + 49\\ &= 49(8P + k + 1), \text{ which is divisible by 49.}\end{aligned}$$

Step 3 By the principle of mathematical induction, the statement is true for all integers $n \geq 0$.

Common mistakes:

- Some students forgot to show the proof for the base value of n ($n = 0$), therefore lost one mark
- Very few students did the proof for divisible by 7 instead of 49
- Very few showed incorrect working for the $n = k + 1$ case, did not write using assumption, hence lost a mark
- Few students used the assumption but made the expression complicated
- Few students forgot to mention that the expression with 49 is an integer, hence lost a mark
-

Overall well done.

Learning Strategies:

To attempt questions on mathematical induction, students need to

- Follow teachers notes closely and write all the four steps in the exam

Question 14 (continues)

(c) A rabbit population of 500 was released on an island. The population growth is
4

modelled by the logistic equation $\frac{dP}{dt} = \frac{P}{10}\left(1 - \frac{P}{2000}\right)$.

Given that $\frac{20000}{P(2000-P)} = 10\left(\frac{1}{P} + \frac{1}{2000-P}\right)$, solve the differential equation

to show that the population P at time t months after introduction is $P = \frac{2000}{1+3e^{-\frac{t}{10}}}$.

$$\frac{dP}{dt} = \frac{P}{10}\left(1 - \frac{P}{2000}\right)$$

$$\frac{dP}{dt} = \frac{P}{10}\left(\frac{2000-P}{2000}\right)$$

$$\frac{dP}{dt} = \frac{P(2000-P)}{20000}$$

$$\frac{20000}{P(2000-P)}\,dP = dt$$

$$\int \frac{20000}{P(2000-P)}\,dP = \int dt$$

$$10\int \left(\frac{1}{P} + \frac{1}{2000-P}\right) dP = \int dt$$

$$\ln|P| - \ln|2000-P| = \frac{t}{10} + C$$

Majority of the students arrived at the above step. Well done.

$$\ln\left|\frac{P}{2000-P}\right| = \frac{t}{10} + C$$

$$\left|\frac{P}{2000-P}\right| = e^{\frac{t}{10}+C}$$

$$\frac{P}{2000-P} = \pm e^{C} e^{\frac{t}{10}}$$

$P = Ae^{\frac{t}{10}}(2000-P),$ where A is a nonzero constant

$$P = 2000Ae^{\frac{t}{10}} - PAe^{\frac{t}{10}}$$

$P(1 + Ae^{\frac{t}{10}}) = 2000Ae^{\frac{t}{10}}$

$$P = \frac{2000Ae^{\frac{t}{10}}}{1 + Ae^{\frac{t}{10}}}$$

Dividing by $Ae^{\frac{t}{10}}$,

$$P = \frac{2000}{\frac{1}{A}e^{-\frac{t}{10}} + 1}$$

$$P = \frac{2000}{Be^{-\frac{t}{10}} + 1}$$ where B is a nonzero constant

Substitute $t = 0, P = 500$,

$$500 = \frac{2000}{B + 1}$$

$1 + B = 4$

$\therefore B = 3$

$$\therefore P = \frac{2000}{1 + 3e^{-\frac{t}{10}}}$$

- few students messed up the expression and did not arrive at the desired step

- few students stopped just one line before the desired expression, I could not award them as it was a "show that" question

Overall well done.

Learning Strategies:

For Logistics questions, try to avoid the bounds on the integration, stick to finding the value of the constant using the given conditions.

Question 14 (continues)

(d) A stone is projected from level ground with initial speed V ms^{-1} at an angle of θ to the horizontal. The maximum height reached by the stone was 8 metres.

(i) By integrating vectors, show that the velocity and displacement of the stone at t seconds are as below. **2**

$$\underset{\sim}{v} = V\cos\theta\,\boldsymbol{i} + (V\sin\theta - gt)\boldsymbol{j}$$

$$\underset{\sim}{r} = Vt\cos\theta\,\boldsymbol{i} + \left(Vt\sin\theta - \frac{g}{2}t^2\right)\boldsymbol{j}$$

$\underset{\sim}{a} = -g\mathbf{j}$

$\underset{\sim}{v} = \int -g\mathbf{j}\,dt$

$\underset{\sim}{v} = -gt\mathbf{j} + \underset{\sim}{C}$

When $t = 0, \underset{\sim}{v} = V\cos\theta\,\mathbf{i} + V\sin\theta\,\mathbf{j}$,

$\underset{\sim}{C} = V\cos\theta\,\mathbf{i} + V\sin\theta\,\mathbf{j}$

$\underset{\sim}{v} = -gt\mathbf{j} + V\cos\theta\,\mathbf{i} + V\sin\theta\,\mathbf{j}$

$\therefore \underset{\sim}{v} = V\cos\theta\,\mathbf{i} + (V\sin\theta - gt)\mathbf{j}$ 1

$\underset{\sim}{r} = \int (V\cos\theta\,\mathbf{i} + (V\sin\theta - gt)\mathbf{j})\,dt$

$\underset{\sim}{r} = Vt\cos\theta\,\mathbf{i} + \left(Vt\sin\theta - \frac{gt^2}{2}\right)\mathbf{j} + \underset{\sim}{D}$

When $t = 0, \underset{\sim}{r} = \underset{\sim}{0}$,

$\underset{\sim}{D} = \underset{\sim}{0}$

$\therefore \underset{\sim}{r} = Vt\cos\theta\,\mathbf{i} + \left(Vt\sin\theta - \frac{gt^2}{2}\right)\mathbf{j}$ 1

- Many students derived the equations for displacement vector and the velocity vector well
- Some students just wrote the integration without finding the values of the constant and since it is a "show that" question, they lost mark(s).

Overall, well done.

(ii) Prove that the horizontal range of the stone is $\sqrt{\dfrac{64}{g}(V^2 - 16g)}$. **3**

Maximum height of 8 when $\dot{y} = 0$

$V \sin\theta - gt = 0$

$t = \dfrac{V\sin\theta}{g}$ (1)

- Many students arrived at this step successfully.

$y = Vt\sin\theta - \dfrac{gt^2}{2}$

$8 = V\sin\theta\left(\dfrac{V\sin\theta}{g}\right) - \dfrac{g}{2}\left(\dfrac{V\sin\theta}{g}\right)^2$

$8 = \dfrac{V^2\sin^2\theta}{g} - \dfrac{V^2\sin^2\theta}{2g}$

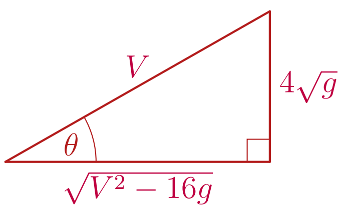
$16g = 2V^2\sin^2\theta - V^2\sin^2\theta$

$16g = V^2\sin^2\theta$

$\sin^2\theta = \dfrac{16g}{V^2}$

$\sin\theta = \dfrac{4\sqrt{g}}{V}$ $(\sin\theta > 0$ since θ is acute)

- Some students arrived at this step, but did algebraic mistakes, and as a result, only few could arrive at this step



$$\therefore \cos\theta = \frac{\sqrt{V^2 - 16g}}{V}$$

$$\text{Time of flight} = \frac{2V\sin\theta}{g} \qquad \big(\text{From (1)}\big)$$

$$= \frac{8\sqrt{g}}{g}$$

$$= \frac{8}{\sqrt{g}}$$

$$x = Vt\cos\theta$$

$$= V\left(\frac{8}{\sqrt{g}}\right)\left(\frac{\sqrt{V^2 - 16g}}{V}\right)$$

$$= \frac{8\sqrt{V^2 - 16g}}{\sqrt{g}}$$

$$= \sqrt{\frac{64(V^2 - 16g)}{g}}$$

$$\therefore x = \sqrt{\frac{64}{g}(V^2 - 16g)}$$

1

- Very few students arrived at this step.
- Poor attempt
-

Learning Strategies:

Students are advised to challenge themselves with questions which requires an in depth knowledge of the topic(s)